ON THE ALMOST SURE CENTRAL LIMIT THEOREM FOR THE ELEPHANT RANDOM WALK

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In this paper we present versions of the Almost Sure Central Limit Theorem for both scalar and multi-dimensional Elephant Random Walk based in the Almost Sure Central Limit for Martingales. In addition, convergence to even moments of Gaussian distribution also will be discussed.

Keywords: Elephant random walk, almost sure central limit theorem, weak convergence.

1. Introduction

Roughly speaking, the Almost Sure Central Limit Theorem (ASCLT) asserts that, given a sequence of independent and identically distributed random variables; (Y_n), we have the weak convergence of certain sequence of Probability measures associated to the partial sums $C_n := Y_1 + \cdots Y_n$ into a Gaussian measure for almost every trajectory (or sample path). In practice, what it is observed from a discrete time stochastic process is a sequence of real numbers: a sample path, hence the ASCLT provides us a tool that induces Probability measures for every trajectory which in almost every case (i.e., with Probability one) posses certain asymptotic behaviour. In other words, the Probability of observing a sample path such that, referenced convergence is observed is one. Its first version was proved by Brossamler [6] and Schatte [10] and in its present form by Lacey and Phillipp [9].

Formally, one of the simplest forms of this theorem is as follows:

Let (Y_n) be a sequence of independent and identically distributed square integrable random variables, such that, for all $n \geq 1$, $\mathbb{E}[Y_n] = 0$ and $\mathbb{E}[Y_n^2] = 1$, then the probability of the event where the sequence of Probability measures

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{C_k/\sqrt{k}} \tag{1}$$

converges weakly to the standard Gaussian law is one (i.e., almost surely or a.s. for short).

Here, $\delta(x)$ is the Dirac mass at point x. We may see that, by observing a single trajectory of process (C_n), we may conclude the weak convergence of measures given by (1).

2. The scalar ERW

The elephant random walk (proposed in [11]); denoted by (S_n) behaves as follow:

1. At time $n = 0$ the elephant is at the origin; i.e., $S_0 = 0$.

2. At time $n = 1$, the elephant decides to move one unit right with probability $q \in (0, 1)$ or one unit left with probability $1 - q$. If X_1 represents such movement, then $S_1 = X_1$ has the Rademacher distribution with parameter q. In general, let X_n be the movement of the elephant at time n.

3. At any time $n \geq 2$, the elephant chooses uniformly at random some point in the past; let us say $1 \leq k \leq n - 1$. Then, its following movement; X_n, will be equal to X_k (the movement made at chosen time k) with probability $p \in [0, 1]$ and equal to $-X_k$ with probability $1-p$. Parameter p is know as the memory parameter of the ERW.

4. Hence, the position of the elephant at time $n \geq 1$ is given by
$$S_n = S_{n-1} + X_n. \tag{2}$$

In order to introduce the martingale approach for the ERW, let $(\mathcal{F}_n)$ be the increasing sequence of σ-algebras $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$; i.e. $\mathcal{F}_n$ represents the elephant's knowledge up to time n. Then, it may be easily found [2,7] that conditional expected movement and position of the elephant at any time are; respectively, given by

$$\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] = (2p-1)\frac{S_n}{n} \quad \text{a.s.} \tag{3}$$

and

$$\mathbb{E}\left[S_{n+1}|\mathcal{F}_n\right] = \gamma_n S_n \quad \text{a.s., where} \quad \gamma_n = \frac{n+2p-1}{n}. \tag{4}$$

Main results of this sections will be supported by results on real martingales. Hence, let us consider the discrete time scalar martingale [2] (M_n), given for $n \geq 0$, by $M_n = a_n S_n$, where $a_1 = 1$ and, for $n \geq 2$

$$a_n = \prod_{k=1}^{n-1} \gamma_k^{-1} = \frac{\Gamma(n)\Gamma(2p)}{\Gamma(n+2p-1)}, \tag{5}$$

where Γ stands for the Euler Gamma function.

The study of asymptotic properties of the scalar ERW is usually done in 3 regimes:

1. If $p \in [0, 3/4)$ then we are in the diffusive regime.
2. If $p = 3/4$ then the regime is the critical one, and
3. If $p \in (3/4, 1]$ then the regime is called superdiffusive.

This paper deals only with diffusive an critical regimes in both scalar and multi-dimensional framework.

2.1. The diffusive regime.

In this subsection, the diffusive regime for the scalar ERW will be explored; in other words, we will consider that $p \in [0, 3/4)$.

Our first result deals with a version of the ASCLT for the position of the ERW

If memory parameter lies in $[0, 3/4)$, then we have the following almost sure convergence of empirical measures

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{S_k/\sqrt{k}} \Longrightarrow N\left(0, \frac{1}{3-4p}\right) \quad \text{a.s.} \tag{6}$$

In a complementary way, we provide the almost sure convergence to the even moments of the Gaussian$(0, \frac{1}{3-4p})$ distribution.

If memory parameter lies in $[0, 3/4)$, then the following almost sure convergence holds

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{S_k^{2r}}{k^{r+1}} \to \frac{(2r)!}{2^r(3-4p)^r r!} \quad \text{a.s.} \tag{7}$$

We may find out that, if $r = 1$ in Theorem 2.1 then we have the Quadratic Strong Law obtained in Theorem 3.2 of [2].

2.2. The critical regime.

We will investigate now, analogue results to the ones presented in previous section for the critical regime; i.e., when memory parameter $p = 3/4$.

Firstly, we present the corresponding version of the Almost Sure Central Limit Theorem

If memory parameter is equal to $3/4$, then we have the almost sure convergence

$$\frac{1}{\log\log n}\sum_{k=2}^{n}\frac{1}{k\log k}\delta_{S_k/\sqrt{k\log k}} \Longrightarrow N(0,1) \tag{8}$$

We also have the almost sure convergence to the even moments of the Standard Gaussian distribution

If $p = 3/4$, then

$$\frac{1}{\log\log n}\sum_{k=2}^{n}\frac{S_k^{2r}}{(k\log k)^{r+1}} \to \frac{(2r)!}{2^r r!} \quad \text{a.s.} \tag{9}$$

Again, we observe that, if $r = 1$ in Theorem 2.4 then we have the Quadratic Strong Law obtained in Theorem 3.5 of [2].

3. The Multi-dimensonal Elephant Random Walk (MERW)

Let us deal with the multi-dimensional version of the ERW. In this framework, we will consider that the state space of the MERW is $\mathbb{Z}^D$. The behavior of the MERW is as follows:

1. At time $n = 0$ the elephant is at the origin; i.e., $S_0 = 0$.

2. At time $n = 1$, the elephant decides to move one unit in any of the $2D$ directions defined by the axes with identical probability $(1/2D)$.

3. For any time $n \geq 1$, let X_n be the movement of the elephant at time $n \geq 0$. Hence, at any time $n \geq 2$, the elephant chooses uniformly at random some point in the past; let us say $1 \leq k \leq n-1$. Then, its following movement; X_n, will be equal to X_k (the movement made at chosen time k) with probability $p \in [0, 1]$ and equal to any other of the $2D - 1$ directions with identical probability $(1-p)/(2D-1)$.

4. Hence, the position of the elephant at time $n \geq 1$ is given by
$$S_n = S_{n-1} + X_n. \tag{10}$$

5. At any time $n \geq 1$; with k as earlier, the step of the elephant is given by
$$X_{n+1} = A_n X_k. \tag{11}$$
where
$$A_n = \begin{cases} I_D & \text{with probability } p \\ -I_D & \text{with probability } \frac{1-p}{2D-1} \\ J_D & \text{with probability } \frac{1-p}{2D-1} \\ -J_D & \text{with probability } \frac{1-p}{2D-1} \\ & \vdots \\ J_D^{D-1} & \text{with probability } \frac{1-p}{2D-1} \\ -J_D^{D-1} & \text{with probability } \frac{1-p}{2D-1} \end{cases},$$
where I_D is the identity matrix of order D and
$$J_D = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

As can be seen, there exist; naturally, some complications arose by the change in dimension:

1. The number of directions in which the elephant may move at any time is $2D$.
2. Once the point in the past has been chosen; let us say k, its relation with next step is more complicated than in its scalar version.
3. Critical value associated with memory parameter is now
$$p_D = \frac{2D+1}{4D},$$
which is equal to $3/4$ if $D = 1$.
4. However, conditional expected movements and positions have similar expression to the corresponding ones in the scalar case [2]:
$$\mathbb{E}\left[X_{n+1} | \mathcal{F}_n\right] = \frac{a}{n} S_n \quad \text{a.s.} \tag{12}$$
and
$$\mathbb{E}\left[S_{n+1} | \mathcal{F}_n\right] = \eta_n S_n \quad \text{a.s.}. \tag{13}$$
where $a = \frac{2Dp-1}{2D-1}$ and $\eta_n = 1 + \frac{a}{n}$.

In this section we will exploit the theory for multi-dimensional martingales given that, process (M_n) defined by $M_n = b_n S_n$ where $b_0 = 1$, $b_1 = 1$ and, for $n \geq 2$,
$$b_n = \prod_{k=1}^{n-1} \eta_k^{-1} = \frac{\Gamma(n)\Gamma(a+1)}{\Gamma(n+a)}, \tag{14}$$
is a discrete time martingale in $\mathbb{R}^D$ [2].

3.1. The diffusive regime.

Let us consider now the multi-dimensional diffusive regime and provide the ASCLT for the position of the elephant as well as the corresponding convergence to even moments of Gaussian distribution.

If memory parameter lies in $[0, p_D)$, then we have the following almost sure convergence of empirical measures
$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{S_k/\sqrt{k}} \Longrightarrow N\left(0, \frac{1}{D(1-2a)} I_D\right) \tag{15}$$

In addition, we have the following almost sure convergence.

If $p \in [0, p_D)$ then, for any $u \in \mathbb{R}^D$ and $r \in \mathbb{N}$ we have that
$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k^{r+1}} \left(u^t S_k S_k^t u\right)^r = \frac{\|u\|^{2r} (2r)!}{(2|1-2a|D)^r r!} \quad \text{a.s.} \tag{16}$$

We observe that almost sure limit in (16) corresponds to the moment of order $2r$ of the $N\left(0, \frac{\|u\|^2}{|1-2a|D}\right)$ distribution.

3.2. The critical regime.

We explore now, the critical regime $p = p_D$ by providing the corresponding version of the ASCLT and the almost sure convergence to even moments of Gaussian distribution.

If memory parameter is equal to the critical value p_D then, we have the following ASCLT:
$$\frac{1}{\log\log n} \sum_{k=1}^{n} \frac{1}{k\log k} \delta_{S_k/\sqrt{k\log k}} \Longrightarrow N\left(0, \frac{1}{D} I_D\right) \quad \text{a.s.} \tag{17}$$

Corresponding result for almost sure convergence of moments is as follows:

If $p = p_D$ then, for any $u \in \mathbb{R}^D$ and $r \in \mathbb{N}$ we have that
$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} \left(\frac{1}{k\log k}\right)^{r+1} \left(u^t S_k S_k^t u\right)^r = \frac{\|u\|^{2r} (2r)!}{(2D)^r r!} \quad \text{a.s.} \tag{18}$$

4. Conclusion

Thanks to martingale theory, in particular, results related to the Almost Sure Central Limit Theorem, it was possible to obtain versions of this theorem concerning with the position of the Elephant random walk in both scalar and multi-dimensional framework, as well as the almost sure convergence to the even moments of Gaussian distribution. Such results lead us; by observing a single path of the ERW, to dilucidate the mean value of any even power of the ERW (properly scaled) and to be able of making asymptotic statistical conclusions for this stochastic process.

In the spirit of paper due to Berkes and Csáki [5], we observe that variances in (6), (8), (15) and (17) coincide with those in Central Limit Theorems provided by Bercu [2], Bercu and Laulin [4], and related with asymptotic normality proposed by Coletti et al. [7]. Hence, ASCLT provided in this paper may be considered as their almost sure versions. In addition, as was said at the Introduction, this situation leads us to conjecture that results of Berkes and Csáki may hold for the Martingale framework.

The superdiffusive regime was not considered in this paper, however, we may find references where it is concluded that limiting distribution of the ERW in the scalar case is not Gaussian [2] [8]. In the multi-dimensional framework it was demonstrated the almost sure convergence as well as the mean square convergence of the Multi-dimensional ERW to a non-degenerate random vector [4], but to the best of our knowledge it is not possible to find results on the weak convergence of the Multi-dimensional ERW.

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